Observability of Linear Hybrid Systems*

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Abstract. We analyze the observability of the continuous and discrete states of a class of continuous-time linear hybrid systems. We derive necessary and sufficient conditions that the structural parameters of the model must satisfy in order for filtering and smoothing algorithms to operate correctly. Our conditions are simple rank tests that exploit the geometry of the observability subspaces generated by the output of a linear hybrid system. We also derive weaker rank conditions that guarantee the uniqueness of the reconstruction of the state trajectory from a specific output, even when the hybrid system is unobservable.

1 Introduction

Observability refers to the study of the conditions under which it is possible to uniquely infer the state of a dynamical system from measurements of its output. When the dynamics of the system are linear, it is well known that the observability problem can be reduced to that of analyzing the rank of the so-called *observability matrix*. This is the well known Popov-Belevic-Hautus rank test for linear systems [5]. The concept of observability has also been extended to the case of systems with nonlinear and smooth dynamics [**?**]. We refer interested readers to [4] and references therein for a recent comparison of different definitions of observability and their equivalence. As for the case of hybrid systems, most of the previous work has concentrated on the areas of modeling, stability, controllability and verification (See previous workshop proceedings). Relatively little attention, however, has been devoted to the study of the observability of both the continuous and discrete states of a hybrid system.

To the best of our knowledge, the first attempt to characterize the observability of hybrid systems can be found in [8], although the condition given does not offer much more insight than the definition itself. Observability has also been addressed recently in [6], which gives an unusual condition in terms of the existence of a discrete state trajectory. This condition pertains to systems where the discrete state is controlled, rather than evolving out of its own dynamics. [7]

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gives conditions for a particular class of linear time-varying systems where the system matrix is a linear combination of a basis with respect to time-varying coefficients. [3] addresses observability and controllability for switched linear systems with known and periodic transitions. [1] proposes the notion of incremental observability of a hybrid system, which requires the solution of a mixed-integer linear program in order to be tested.

In this paper, we study the observability of a class of linear hybrid systems known as jump- (or switched-) linear systems, *i.e.,* systems whose evolution is determined by a collection of linear models with *continuous state* $x_t \in \mathbb{R}^n$ connected by switches among a number of *discrete states* $\lambda_t \in \{1, 2, \dots, N\}$. In Section 2 we introduce a notion of observability for linear hybrid systems. We define the obsevability index of a jump linear system and use it to derive rank conditions that the structural parameters of the model must satisfy in order for filtering and smoothing algorithms to operate correctly. We show that state trajectory is observable if and only if the pairwise intersection of different observable subspaces is trivial. We also show that the switching times are observable if and only if the difference of any pair of observability matrices is nonsingular. The rank conditions we derive are simpler than their discrete-time counterparts [9] and can be thought of as an extension of the Popov-Belevic-Hautus rank test for linear systems [5]. Our conditions only depend on the geometry of the observability subspaces, and therefore they are applicable also to the case of hybrid models where the switching mechanism depends on the continuous state. In Section 3 we derive weaker rank conditions that guarantee the uniqueness of the reconstruction the state trajectory from a specific output, even if the linear hybrid system is unobservable. Our conditions simply require that the given output switches at least once in the given time interval. Section 5 concludes.

2 Observability of Linear Hybrid Systems

We consider a class of continuous-time hybrid systems known as jump linear systems, *i.e.,* systems whose evolution is determined by a collection of linear models with *continuous state* $x_t \in \mathbb{R}^n$ connected by switches of a number of *discrete states* $\lambda_t \in \{1, 2, \dots, N\}$. The evolution of the continuous state x_t is described by the linear system

$$\dot{x}_t = A(\lambda_t)x_t \tag{1}$$

$$y_t = C(\lambda_t)x_t \tag{2}$$

where $A(k) \in \mathbb{R}^{n\times n}$ and $C(k) \in \mathbb{R}^{p\times n}$, for $k \in \{1, 2, \dots, N\}$. The evolution of the discrete state λ_t can be modeled, for instance, as an irreducible Markov chain governed by the transition map π, $P(\lambda_{t+1}|\lambda_t) = \pi_{t+1,t}$ or, as we do here, as a deterministic but unknown input that is piecewise constant, right-continuous and finite-valued[1]. Furthermore, we assume that the hybrid system admits no Zeno

[1] Most of the literature on hybrid systems restricts the switching mechanism of the discrete state to depend on the value of the continuous state. While this is gener-

executions. More specifically, we assume that the switching times $\{t_k, k \geq 1\}$ are separated by at least $\tau > 0$; that is, we assume that $t_{k+1} - t_k \geq \tau > 0$. Having a minimum separation τ between consecutive switches is not a strong assumption to make, since τ can be arbitrarily small, as long as it is constant and positive.

Given a linear hybrid system $\Sigma = \{A(k), C(k); k = 1 \dots N\}$, we focus our attention on how to infer the state of the system $\{x_t, \lambda_t\}$ from the output $\{y_t\}$. The simplest instance of this problem can be informally described as follows. Assume that we are given the model parameters $A(\cdot), C(\cdot)$ and that Σ evolves starting from an (unknown) initial condition (x_{t_0}, λ_{t_0}). Given the output $\{y_t\}$ in the interval $[t_0, t_0+T]$, is it possible to reconstruct the continuous state trajectory x_t and the discrete state trajectory λ_t uniquely?

If the sequence of discrete states $\lambda_{t_0}, \lambda_{t_1}, \dots, \lambda_{t_0+T}$ is known, then the output of the system between two consecutive jumps can be written explicitly in terms of the model parameters $A(\cdot), C(\cdot)$, and the initial value of the continuous state x_{t_0} as:

$$y_t = \begin{cases} C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} & t \in [t_0, t_1) \\ C(\lambda_{t_1}) e^{A(\lambda_{t_1})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0} & t \in [t_1, t_2) \\ \vdots & \vdots \end{cases}$$

We thus propose the following notions of indistinguishability and observability:

Definition 1 (Indistinguishability). *We say that the states* $\{x_{t_0}, \lambda_t\}$ *and* $\{\bar{x}_{t_0}, \bar{\lambda}_t\}$ *are* **indistinguishable** *on the interval* $t \in [t_0, t_0+T]$ *if the corresponding outputs in free evolution* $\{y_t\}$ *and* $\{\bar{y}_t\}$ *are equal. We use* $\{x_{t_0}, \lambda_{t_0}, \dots, \lambda_{t_0+T}\}$ *instead of* $\{x_{t_0}, \lambda_t\}$ *to denote the state when the switching times are known. We denote the set of states which are indistinguishable from* $\{x_{t_0}, \lambda_t\}$ *as* $\mathcal{I}(x_{t_0}, \lambda_t)$.

Definition 2 (Observability). *We say that a state* $\{x_{t_0}, \lambda_t\}$ *is observable on* $t \in [t_0, t_0+T]$ *if* $\mathcal{I}(x_{t_0}, \lambda_t) = \{x_{t_0}, \lambda_t\}$. *When any admissible state is observable, we say that the model* Σ *is* **observable**.

2.1 Observability of the initial state

We first analyze the conditions under which we can determine x_{t_0} and $\lambda_t = \lambda_{t_0}$ for $t \in [t_0, t_1)$ uniquely, *i.e.,* before a switch occurs. We have that $\{x_{t_0}, \lambda_{t_0}\}$ is indistinguishable from $\{\bar{x}_{t_0}, \bar{\lambda}_{t_0}\}$ if and only if

$$C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} = C(\bar{\lambda}_{t_0}) e^{A(\bar{\lambda}_{t_0})(t-t_0)} \bar{x}_{t_0} \qquad \text{for } t \in [t_0, t_1). \tag{3}$$

ally sensible in the study of stability, it is a significant restriction to impose in the context of filtering and identification. Our model is more general, as it imposes no restriction on the mechanism that governs the transitions between discrete states. The conditions we derive, therefore, also apply to systems with state-dependent transitions.

After expanding both sides in Taylor series about t_0, the indistinguishability condition can be written as

$$y_{t_0}^{(k)} = C(\lambda_{t_0})A(\lambda_{t_0})^k x_{t_0} = C(\bar{\lambda}_{t_0})A(\bar{\lambda}_{t_0})^k \bar{x}_{t_0} \qquad \text{for } k \geq 0. \tag{4}$$

If we let $\mathcal{O}_\infty(\lambda_{t_0})$ and $\mathcal{O}_\infty(\bar{\lambda}_{t_0})$ be the infinite-dimensional extended observability matrices of the pairs $(A(\lambda_{t_0}), C(\lambda_{t_0}))$ and $(A(\bar{\lambda}_{t_0}), C(\bar{\lambda}_{t_0}))$, respectively, then the indistinguishability condition can be compactly written as

$$\mathcal{O}_\infty(\lambda_{t_0})x_{t_0} = \mathcal{O}_\infty(\bar{\lambda}_{t_0})\bar{x}_{t_0}. \tag{5}$$

Therefore, the initial state $\{x_{t_0}, \lambda_{t_0}\}$ is observable if and only if the observability subspaces do not intersect; that is, if $\text{rank}([\mathcal{O}_\infty(\lambda_{t_0}) \;\; \mathcal{O}_\infty(\bar{\lambda}_{t_0})]) = 2n$. It turns out that, as in the linear systems case, we can restrict our attention to finite-dimensional observability matrices, because the *extended joint observability matrix* $\mathcal{O}_\infty(k,k') \triangleq [\mathcal{O}_\infty(k) \; \mathcal{O}_\infty(k')]$ equals the extended observability matrix of the $2n$-dimensional system defined by:

$$A(k,k') = \begin{bmatrix} A(k) & 0 \\ 0 & A(k') \end{bmatrix} \quad C(k,k') = \begin{bmatrix} C(k) \; C(k') \end{bmatrix}.$$

Hence, we define the *joint observability index* of systems k and k' as the minimum integer $\nu(k,k')$ such that the rank of the finite-dimensional joint observability matrix $\mathcal{O}_j(k,k') \triangleq [\mathcal{O}_j(k) \; \mathcal{O}_j(k')]$, where

$$\mathcal{O}_j(k) = [C(k)^T \; (C(k)A(k))^T \; \cdots \; (C(k)A(k)^{j-1})^T]^T, \tag{6}$$

stops growing. Thus, we can rephrase the indistinguishability condition in terms of the largest joint observability index $\nu \triangleq \max_{k \neq k'}\{\nu(k,k')\} \leq 2n$ as:

$$\mathcal{Y}_\nu(t_0) \triangleq \begin{bmatrix} y_{t_0} \\ \dot{y}_{t_0} \\ \vdots \\ y_{t_0}^{(\nu-1)} \end{bmatrix} = \mathcal{O}_\nu(\lambda_{t_0})x_{t_0} = \mathcal{O}_\nu(\bar{\lambda}_{t_0})\bar{x}_{t_0}. \tag{7}$$

¿From this equation we derive the following condition on the observability of the initial state $\{x_{t_0}, \lambda_{t_0}\}$:

Lemma 1 (Observability of the initial state). *If $t_1 - t_0 \geq \tau > 0$, then the initial state $\{x_{t_0}, \lambda_{t_0}\}$ is observable if and only if for all $k \neq k' \in \{1, \ldots, N\}$ we have* $\text{rank}([\mathcal{O}_\nu(k) \; \mathcal{O}_\nu(k')]) = 2n$*. Furthermore, the initial state is given by:*

$$\lambda_{t_0} = \{k : \text{rank}([\mathcal{O}_\nu(k) \;\; \mathcal{Y}_\nu(t_0)]) = n\} \qquad x_{t_0} = \mathcal{O}_\nu(\lambda_{t_0})^\dagger \mathcal{Y}_\nu(t_0). \tag{8}$$

We illustrate the applicability of Lemma 1 with the following example:

Example 1 (Two observable linear systems give an unobservable hybrid system). Consider a one-dimensional linear hybrid system composed of the two linear systems:

$$\begin{array}{ll} \dot{x} = 0 & \dot{x} = 0 \\ y = c_1 x & y = c_2 x \end{array}, \tag{9}$$

where $c_1 \neq 0$, $c_2 \neq 0$ and $c_1 \neq c_2$. We observe that the initial state of each linear system is observable, but the initial state of the linear hybrid system is not: One can set the initial condition of system 1 to x_0 and the initial condition of system 2 to $c_1 x_0 / c_2$ and obtain identical outputs. That is, states $(x_0, 1)$ and $(c_1 x_0 / c_2, 2)$ are indistinguishable. Notice that in this example the rank-$2n$ condition is violated, because $\text{rank}[c_1 \; c_2] = 1 < 2$.

Remark 1 (Observability subspaces). Notice that the rank-$2n$ condition implies that each linear system $(A(k), C(k))$ must be observable, because we must have $\text{rank}(\mathcal{O}_\nu(k)) = n$ for all $k \in \{1, \ldots, N\}$. In addition, the rank-$2n$ condition implies that the intersection of the observability subspaces of each pair of linear systems has to be trivial. In fact, the set of unobservable states can be directly obtained from the intersection of the observability subspaces. One could therefore introduce a notion of distance between models using the angles between the observability spaces, similarly to [2].

2.2 Observability of the first transition

Lemma 1 provides conditions for the observability of the initial state $\{x_{t_0}, \lambda_{t_0}\}$. We are now interested in the observability of $\{x_{t_0}, \lambda_t\}$ for $t \in [t_0, t_1]$. Since λ_t is a piecewise constant function, we only need to concentrate on the conditions under which the first transition, t_1, can be uniquely determined. We have

$$y_t = \begin{cases} C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} & t \in [t_0, t_1) \\ C(\lambda_{t_1}) e^{A(\lambda_{t_1})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0} & t \in [t_1, t_2) \end{cases} \tag{10}$$

and want to determine if it is possible that

$$y_t = \begin{cases} C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_0)} x_{t_0} & t \in [t_0, \bar{t}_1) \\ C(\lambda_{\bar{t}_1}) e^{A(\lambda_{\bar{t}_1})(t-\bar{t}_1)} e^{A(\lambda_{t_0})(\bar{t}_1-t_0)} x_{t_0} & t \in [\bar{t}_1, t_2) \end{cases} \tag{11}$$

where $\lambda_{\bar{t}_1} = \lambda_{t_1}$, because the discrete state sequence is uniquely determined thanks to the rank-$2n$ condition. Without loss of generality, assume that $\bar{t}_1 > t_1$ and consider the output y_t in the interval $[t_1, \bar{t}_1)$. We observe that t_1 is indistinguishable if and only if for $t \in [t_1, \bar{t}_1)$

$$C(\lambda_{t_0}) e^{A(\lambda_{t_0})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0} = C(\lambda_{t_1}) e^{A(\lambda_{t_1})(t-t_1)} e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0}. \tag{12}$$

After expanding both sides in Taylor series about t_1, the indistinguishability condition can be written as

$$y_{t_1}^{(k)} = C(\lambda_{t_0}) A(\lambda_{t_0})^k x_{t_1} = C(\lambda_{t_1}) A(\lambda_{t_1})^k x_{t_1} \qquad \text{for } k \geq 0. \tag{13}$$

As before, then the indistinguishability condition can be compactly written in terms of the extended observability matrices as:

$$\mathcal{O}_\nu(\lambda_{t_0})x_{t_1} = \mathcal{O}_\nu(\lambda_{t_1})x_{t_1}. \tag{14}$$

Hence, t_1 is indistinguishable when the difference between the observability matrices $\mathcal{O}_\nu(\lambda_{t_0}) - \mathcal{O}_\nu(\lambda_{t_1})$ is singular. Since this could happen for any pair of observability matrices, in order for t_1 to be observable, we need to ensure that the difference of any pair of observability matrices is nonsingular.

Lemma 2 (Observability of the first transition). *If $t_1 - t_0 \geq \tau > 0$, then the first transition is observable if and only if for all $k \neq k' \in \{1, \ldots, N\}$ we have* $\text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$*. Furthermore, the first transition can recovered as the time instance at which the output y_t is not $\mathcal{C}^\infty$, i.e.,*

$$t_1 = \min\{t > t_0 : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\}. \tag{15}$$

Remark 2 (Continuous reset map). Notice that if the continuous reset is different from the identity map, then the switching times can be found by looking at the discontinuities of y_t directly, with no need for higher-order derivatives of y_t.

Remark 3 (Unobservable subspaces). Notice that if $[\mathcal{O}_\nu(k); \mathcal{O}_\nu(k')]x = 0$, then $\mathcal{O}_\nu(k)x = \mathcal{O}_\nu(k')x = 0$, hence $(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k'))x = 0$. Thus the rank-$n$ condition $\text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$ implies that the unobservable subspaces of any pair of observability matrices must not intersect. While this observation is irrelevant for the observability of a linear hybrid system, because each linear system has to be observable (See Remark 1), it will be quite important for uniquely reconstructing the state trajectory, as we will discuss in Section 3.

2.3 Observability of linear hybrid systems

Once x_{t_0}, λ_{t_0} and t_1 have been determined, we just repeat the process for the remaining jumps. The only difference is that x_{t_k}, $k \geq 1$, will be given. However, since λ_{t_0} is originally unknown, we still need to check the rank-$2n$ condition of Lemma 1 for any pair of extended observability matrices in order for x_{t_0} and λ_{t_0} to be uniquely recoverable. Therefore, since the rank-$2n$ condition $\text{rank}([\mathcal{O}_\nu(k) \ \ \mathcal{O}_\nu(k')]) = 2n$ implies the rank-n condition $\text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$, we have the following theorem on the observability of jump linear systems:

Theorem 1. *If for all $k \geq 0$ we have $t_{k+1} - t_k \geq \tau > 0$ and for all $k \neq k' \in \{1, \ldots, N\}$ we have* $\text{rank}([\mathcal{O}_\nu(k) \ \ \mathcal{O}_\nu(k')]) = 2n$ *then $\{x_{t_0}, \lambda_t\}$ is observable on $t \in [t_0, t_0 + T]$. Furthermore, the state trajectory can be uniquely recovered as:*

$$\lambda_{t_0} = \{k : \text{rank}([\mathcal{O}_\nu(k) \ \ \mathcal{Y}_\nu(t_0)]) = n\} \tag{16}$$

$$x_{t_0} = \mathcal{O}_\nu(\lambda_{t_0})^\dagger \mathcal{Y}_\nu(t_0) \tag{17}$$

$$t_i = \min\{t > t_{i-1} : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\} \tag{18}$$

$$\lambda_{t_i} = \{k : \text{rank}([\mathcal{O}_\nu(k) \ \ \mathcal{Y}_\nu(t_i)]) = n\}. \tag{19}$$

Remark 4 (Observability of discrete-time linear hybrid systems). Notice that the rank conditions of Theorem 1 are simpler than their discrete-time counterparts (See [9]). In discrete time, it is possible that a switch occurs at time t_i but its effect in the output appears some time steps after t_i. In that case, in order to guarantee observability, additional rank constraints need to be imposed, for example the $A(\cdot)$ matrices must be nonsingular and they cannot commute.

Remark 5 (Observability of linear hybrid systems in terms of observability operators). The rank constraints of Theorem 1 can also be expressed in terms of observability operators. For example, let $\mathcal{L}(k)\ :\mathbb{R}^n \rightarrow \mathcal{C}^{\infty}_{[0,\tau]}$ be defined by $x \mapsto y(t) = C(k)\mathrm{e}^{A(k)(t)}x = [\mathcal{L}(k)x](t)$ for $t \in [0,\tau]$. Let us also introduce the adjoint observability operator $\mathcal{L}^*(k)\ : \mathcal{C}^{\infty}_{[0,\tau]} \rightarrow \mathbb{R}^n$ such that $\xi(\cdot) \mapsto x = \int_0^{\tau} \mathrm{e}^{A^T(k)(\tau-s)}C^T\xi(s)\,ds = \mathcal{L}^*(k)\xi$ for $\xi(\cdot) \in \mathcal{C}^{\infty}_{[0,\tau]}$. Then a linear hybrid system is observable if and only if for all $k \neq k' \in \{1,\ldots,N\}$ we have $Range(\mathcal{L}(k)) \cap Range(\mathcal{L}(k')) = 0$ and the map $\mathcal{L}(k) - \mathcal{L}(k')$ is injective. Then one can reconstruct the state trajectory by orthogonally projecting the output onto the range of these observability operators. More specifically, one can determine the initial discrete state λ_0 by looking at k such that $y(t) - \left[\mathcal{L}(k)(\mathcal{L}^*(k)\mathcal{L}(k))^{-1}\mathcal{L}^*(k)y\right](t) = 0 \quad \forall t \in [0,\tau), \quad \tau \leq t_1$. The first switching time t_1 can be determined as the first time instant such that $y(t) - \left[\mathcal{L}(k)(\mathcal{L}^*(k)\mathcal{L}(k))^{-1}\mathcal{L}^*(k)y\right](t) \neq 0, \quad t \geq t_1$. The same argument applies for the subsequent discrete states and switching times. The initial continuous state is then easily obtained.

Remark 6. In the paper we have chosen to state results in terms of derivatives of the output as it is, in our opinion, easier for expositional reasons and also help to make contact with the discrete time case. Nevertheless, when it will come do computations and quantify errors working with grammians may turn out to be more convenient. Just to mention one point, in practice one need to quantify "how far" the systems are and how this affects the estimation of the initial state and the discrete sequence in the presence of "noise". If one considers, for instance, L^2 distances in the output spaces $d_{[0,\tau]}(k) = \int_0^{\tau} \|y(t) - \left[\mathcal{L}(k)(\mathcal{L}^*(k)\mathcal{L}(k))^{-1}\mathcal{L}^*(k)y\right](t)\|^2\,dt$ then a natural way to measure distance between systems is by looking at the subspace angles between observability spaces, as suggested in [2]. In fact, assume $y(t)$ has been generated by system 1 and we measure $d_{[0,\tau]}(2)$ then it holds that $d_{[0,\tau]}(2) \geq \|y\|^2 \sin^2(\theta_{min})$ where θ_{min} is the smallest canonical angle between $Range(\mathcal{L}(1))$ and $Range(\mathcal{L}(2))$. Investigating this questions will be subject of future research and we shall not discuss it further in this paper.

3 State estimation from a particular output

Theorem 1 gives *necessary and sufficient* conditions for the observability of a class of hybrid systems in which the switching mechanism is exogenous. Since the theorem imposes no restriction on the mechanism that governs the transitions between discrete states, the conditions of Theorem 1 remain *sufficient* for linear hybrid systems in which switching mechanism depends on the value of the continuous state, e.g. piecewise affine systems. In fact, there may be cases in which the hybrid system is itself unobservable (in the sense of Definition 2), yet it is possible to uniquely reconstruct the state trajectory for a particular output[2]. Intuitively, this happens when the hybrid system switches from an unobservable state to an observable one, as we illustrate in the following example.

Example 2 (Unique reconstruction from two unobservable linear systems). Consider a two-dimensional linear hybrid system composed of the two linear systems

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \qquad & \dot{x} &= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} x \qquad & y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{aligned} \tag{20}$$

Let $t_0 = 0$, $T = 2$, $x_0 = [0, 1]^T$ and assume that there is a single switch from system 1 to system 2 at time $t_1 = 1$. Then $\nu = 2$,

$$x_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} \quad y_t = \begin{cases} 0 & t \in [0, 1) \\ e^{2t} & t \in [1, 2) \end{cases} \quad \mathcal{O}_2(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \mathcal{O}_2(2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

In this example, both linear systems are unobservable and the initial condition lies in the unobservable subspace of system 1. Also notice that the rank-$2n$ condition is violated, since $\operatorname{rank}([\mathcal{O}_2(1) \;\; \mathcal{O}_2(2)]) = 2 < 4$, thus Lemma 1 does not apply. However, the first transition can be uniquely recovered, because y_t is discontinuous at $t_1 = 1$. Furthermore, one can uniquely reconstruct x_{t_0}, λ_{t_0} and λ_{t_1}, because the unobservable subspace of system 1 is observable for system 2 and vice-versa. We make this more precise in the rest of this Section.

Following the example, in this section we derive condition under which one can uniquely reconstruct the state trajectory of a linear hybrid system given a particular output. We show how this can be done despite the individual linear systems being unobservable or the output of the system being zero during a switching interval.

We start by assuming that we know the number and location of the switching times in the interval $[t_0, t_0 + T]$. According to our discussion in the previous section, this is equivalent to assuming that for all $k \neq k' \in \{1, \dots, N\}$ we have $\operatorname{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$. Now, from the indistinguishability condition

$$\mathcal{Y}_\nu(t_0) = \mathcal{O}_\nu(\lambda_{t_0}) x_{t_0} = \mathcal{O}_\nu(\bar{\lambda}_{t_0}) \bar{x}_{t_0}, \tag{21}$$

[2] Notice that this is impossible for linear systems. A linear system is either observable, in which case one can uniquely reconstruct the state for any given output, or it is unobservable, in which case for any output there are always infinitely many possible state trajectories generating it.

we know that the initial state is indistinguishable whenever the the intersection of any pair of observability subspaces is nontrivial, which happens when

$$\mathrm{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) < \mathrm{rank}(\mathcal{O}_\nu(k)) + \mathrm{rank}(\mathcal{O}_\nu(k')). \tag{22}$$

We thus have the following:

1. If $\mathcal{Y}_\nu(t_0) \neq 0$, then the discrete state λ_{t_0} can be uniquely recovered provided that the intersection of any pair of observability subspaces is trivial. That is, for all $k \neq k' \in \{1, \dots, N\}$ we must have

$$\mathrm{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) = \mathrm{rank}(\mathcal{O}_\nu(k)) + \mathrm{rank}(\mathcal{O}_\nu(k')). \tag{23}$$

   In this case we have

$$\lambda_{t_0} = \{k : \mathrm{rank}([\mathcal{O}_\nu(k)\ \ \mathcal{Y}_\nu(t_0)]) = \mathrm{rank}(\mathcal{O}_\nu(k))\}. \tag{24}$$

   Notice that we do not need $\mathcal{O}_\nu(k)$ to be full rank, hence the rank-$2n$ condition may be violated here.
2. If $\mathcal{Y}_\nu(t_0) \neq 0$ and $t_1 > t_0 + T$, *i.e.,* if there is no switch during the observability window, then the continuous state can be uniquely recovered if and only if each linear system is observable, *i.e.,* if for all $k \in \{1, \dots, N\}$ we have $\mathrm{rank}(\mathcal{O}_\nu(k)) = n$. Let λ_{t_0} be defined as in (24), then we have

$$x_{t_0} = \mathcal{O}_\nu(\lambda_{t_0})^\dagger \mathcal{Y}_\nu(t_0). \tag{25}$$

   This means that, if there is no switch, we *do* need every system to be observable and the rank-$2n$ condition has to be in effect.
3. If $\mathcal{Y}_\nu(t_0) \neq 0$ and $t_1 < t_0 + T$, *i.e.,* if at least one switch occurs during the observability window, then the continuous state may *not* be uniquely recovered from the output in the interval $[t_0, t_1)$, but one may still be able to uniquely recover it from the output on the whole interval $[t_0, t_0 + T]$. Loosely speaking, we need to find a condition such that the part of x_{t_0} that is not observable on $[t_0, t_1)$ becomes observable on $[t_1, t_0 + T]$. For example, imagine that there is only one switch at time t_1. Then we have that

$$\begin{bmatrix} \mathcal{O}_\nu(\lambda_{t_0}) \\ \mathcal{O}_\nu(\lambda_{t_1}) e^{A(\lambda_{t_0})(t_1 - t_0)} \end{bmatrix} x_{t_0} = \begin{bmatrix} \mathcal{Y}_\nu(t_0) \\ \mathcal{Y}_\nu(t_1) \end{bmatrix}. \tag{26}$$

   Therefore, in order to determine x_{t_0} uniquely we need

$$\mathrm{rank} \begin{bmatrix} \mathcal{O}_\nu(\lambda_{t_0}) \\ \mathcal{O}_\nu(\lambda_{t_1}) e^{A(\lambda_{t_0})(t_1 - t_0)} \end{bmatrix} = n. \tag{27}$$

   This rank condition is trivially satisfied, because the null-space of $\mathcal{O}_\nu(\lambda_{t_0})$ is $e^{A(\lambda_{t_0})}$-invariant and we have assumed that $\mathrm{rank}(\mathcal{O}(\lambda_{t_1}) - \mathcal{O}(\lambda_{t_0})) = n$ in order for t_1 to be observable (See Remark 3).
   More generally, if there are j switches, $t_1, t_2, \dots t_j$, on the interval $[t_0, t_0 + T]$, $\mathcal{Y}_\nu(t_i) \neq 0$ for $i = 0, 1, \dots, j$, and the corresponding sequence of discrete

states $\lambda_{t_0}, \lambda_{t_1}, \ldots, \lambda_{t_j}$ can be uniquely recovered similarly to (24), then the initial continuous state x_{t_0} can be uniquely recovered from

$$\begin{bmatrix} \mathcal{O}_\nu(\lambda_{t_0}) \\ \mathcal{O}_\nu(\lambda_{t_1}) e^{A(\lambda_{t_0})(t_1-t_0)} \\ \vdots \\ \mathcal{O}_\nu(\lambda_{t_j}) e^{A(\lambda_{t_{j-1}})(t_j-t_{j-1})} \cdots e^{A(\lambda_{t_0})(t_1-t_0)} \end{bmatrix} x_{t_0} = \begin{bmatrix} \mathcal{Y}_\nu(t_0) \\ \mathcal{Y}_\nu(t_1) \\ \vdots \\ \mathcal{Y}_\nu(t_j) \end{bmatrix}. \tag{28}$$

Notice again that the matrix on the left is full rank thanks to the rank-n condition $\text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$.

4. If $\mathcal{Y}_\nu(t_0) = 0$, then we cannot compute λ_{t_0} from (24). However, the rank constraint in (27) guarantees that $\mathcal{Y}_\nu(t_1) = \mathcal{O}_\nu(\lambda_{t_1}) e^{A(\lambda_{t_0})(t_1-t_0)} x_{t_0} \neq 0$. Therefore, we can solve for λ_{t_1} uniquely similarly to (24). Given λ_{t_1}, the rank-n condition $\text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n$ guarantees that λ_{t_0} can be uniquely determined as

$$\lambda_{t_0} = \{k : \text{rank} \begin{bmatrix} \mathcal{O}_\nu(k) & 0 \\ \mathcal{O}_\nu(\lambda_{t_1}) & \mathcal{Y}_\nu(t_1) \end{bmatrix} = n\}. \tag{29}$$

More generally, whenever the output is zero in an interval $[t_i, t_{i+1})$, *i.e.,* whenever $\mathcal{Y}_\nu(t_i) = 0$, we must have that $\mathcal{Y}_\nu(t_{i+1}) \neq 0$ from which we can uniquely recover $\lambda_{t_{i+1}}$ as in (24). Given $\lambda_{t_{i+1}}$ one can uniquely determine λ_{t_i} as in (29). Then we are back into the situation of step 3 in which the discrete sequence is known, hence x_{t_0} can be uniquely recovered from (28).

We summarize our discussion in the following theorem:

Theorem 2 (Reconstruction of the state trajectory). *Consider a linear hybrid system $\Sigma = \{A(k), C(k); k = 1 \ldots N\}$. We have the following:*

1. ***Observability of the switching times****: If the difference of any pairs of observability matrices is nonsingular; that is, if*

$$\textit{for all } k \neq k' \in \{1, \ldots, N\} \textit{ we have } \text{rank}(\mathcal{O}_\nu(k) - \mathcal{O}_\nu(k')) = n, \tag{30}$$

then the switching times can be uniquely recovered as the time instances at which the output y_t is not $\mathcal{C}^\infty$, that is

$$t_i = \min\{t > t_{i-1} : \mathcal{Y}_\nu(t^-) \neq \mathcal{Y}_\nu(t^+)\} \tag{31}$$

We denote by j the number total number of switches in the interval $[t_0, t_0+T]$.

2. ***Reconstruction of the discrete state trajectory****: If in addition the observability subspaces of any pair of observability matrices do not intersect, that is if for all $k \neq k' \in \{1, \ldots, N\}$ we have that*

$$\text{rank}([\mathcal{O}_\nu(k)\ \mathcal{O}_\nu(k')]) = \text{rank}(\mathcal{O}_\nu(k)) + \text{rank}(\mathcal{O}_\nu(k')), \tag{32}$$

then the discrete state trajectory can be uniquely recovered as follows:

(a) *For the switching times t_i such that $\mathcal{Y}_\nu(t_i) \neq 0$, obtain the discrete state similarly to (24) as*

$$\lambda_{t_i} = \{k : \operatorname{rank}([\mathcal{O}_\nu(k) \;\; \mathcal{Y}_\nu(t_i)]) = \operatorname{rank}(\mathcal{O}_\nu(k))\}. \tag{33}$$

(b) *For the switching times t_i such that $\mathcal{Y}_\nu(t_i) = 0$, the rank constraint (30) guarantees that $\mathcal{Y}_\nu(t_{i+1}) \neq 0$, thus $\lambda_{t_{i+1}}$ can be computed from (33). Then λ_{t_i} can be computed similarly to (29) as*

$$\lambda_{t_i} = \{k : \operatorname{rank}\begin{bmatrix} \mathcal{O}_\nu(k) & 0 \\ \mathcal{O}_\nu(\lambda_{t_{i+1}}) & \mathcal{Y}_\nu(t_{i+1}) \end{bmatrix} = n\}. \tag{34}$$

3. ***Reconstruction of the initial continuous state**: Under the conditions stated before, the initial value of the continuous state can be uniquely recovered as:*

$$x_{t_0} = \begin{bmatrix} \mathcal{O}_\nu(\lambda_{t_0}) \\ \mathcal{O}_\nu(\lambda_{t_1})e^{A(\lambda_{t_0})(t_1-t_0)} \\ \vdots \\ \mathcal{O}_\nu(\lambda_{t_j})e^{A(\lambda_{t_{j-1}})(t_j-t_{j-1})}\cdots e^{A(\lambda_{t_0})(t_1-t_0)} \end{bmatrix}^{\dagger} \begin{bmatrix} \mathcal{Y}_\nu(t_0) \\ \mathcal{Y}_\nu(t_1) \\ \vdots \\ \mathcal{Y}_\nu(t_j) \end{bmatrix}. \tag{35}$$

Example 3 (Unique reconstruction from two unobservable hybrid systems). Consider the two-dimensional linear hybrid model of Example 2, where $t_0 = 0$, $t_1 = 1$, $T = 2$, $x_0 = [0, 1]^T$ and

$$\begin{array}{ll} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x & \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} x \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x & y = \begin{bmatrix} 0 & 1 \end{bmatrix} x. \end{array} \qquad x_t = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} \qquad y_t = \begin{cases} 0 & t \in [0, 1) \\ e^{2t} & t \in [1, 2) \end{cases}. \tag{36}$$

In this example we have $\nu = 2$,

$$\mathcal{O}_2(1) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{O}_2(2) = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathcal{Y}_2(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathcal{Y}_2(1) = \begin{bmatrix} e^2 \\ 2e^2 \end{bmatrix}. \tag{37}$$

In this example, both linear systems are unobservable and the initial condition lies in the null-space of $\mathcal{O}_2(1)$. Also notice that the rank-$2n$ condition is violated, since $\operatorname{rank}([\mathcal{O}_2(1) \;\; \mathcal{O}_2(2)]) = 2 < 4$, thus Lemma 1 does not apply. However, we have $\operatorname{rank}(\mathcal{O}_2(1) - \mathcal{O}_2(2)) = 2$, thus t_1 can be uniquely recovered, because y_t is discontinuous at $t_1 = 1$. Also $\operatorname{rank}([\mathcal{O}_2(1) \;\; \mathcal{O}_2(2)]) = \operatorname{rank}(\mathcal{O}_2(1)) + \operatorname{rank}(\mathcal{O}(2)) = 1 + 1 = 2$, thus $\lambda_{t_1} = 2$ can be uniquely recovered, because $\operatorname{rank}([\mathcal{O}_2(1) \; \mathcal{Y}_2(1)]) = 2 \neq 1$ while $\operatorname{rank}([\mathcal{O}_2(2) \; \mathcal{Y}_2(1)]) = 1 = 1$. Similarly, we can uniquely estimate $\lambda_{t_0} = 1$ and $x_{t_0} = [0, 1]^T$.

4 Observability in the Presence of Inputs

The analysis we have carried out in the first part of the paper is limited to the case where the system evolves, unexcited, starting from some initial state.

The conditions we have derived, therefore, involve only the A and C matrices. This is of course only part of the story. In fact, there could be at least one main objection to this approach. The observation of the discrete state could be seen as a classification or better a system identification problem. In fact one has to infer, from data, which system have generated them among a finite class. Therefore we could formalize it as system identification problem where the model class is finite. The main difference here is that one cannot use asymptotics as, by assumptions, the systems switches from one another in finite time. As it is well known inputs plays a fundamental role in system identification and notions such as "persistency of excitation" are fundamental in the field. This basically refers to the property of the input of "moving enough" the systems guaranteeing that difference shows up in the output allowing one to tell them apart.

Let us consider for instance a very simple example where $N = 2$ and the two systems have $i$dentical A and C matrices, but different input-to-state coefficients (say for instance that $B(1) = 2B(2)$. Assume the system is excited with white gaussian noise. Of course it is rather intuitive and easy to prove that as the discrete state jump from 1 to 2, the variance of the state increases and so does the output. Therefore one would be able to detect the jump even though the A, C part are identical (and therefore not observable according to our definitions).

This very simple discussion should warn us that a sensible definition of "observability" should instead include also the input matrices (B, D).

At this point one could be tempted to give observability conditions in term of covariances of the output but again this does not seem to be a very useful one for at least two reasons. First, one will never be able to compute (good approximations of) stationary covariances from finite sequences of data (i.e. from data $a$mong switches). Second, well known results [**?**] show that the output covariance of a Jump Markov Linear System can be realized with a finite dimensional ARMA model, therefore covariance data are NOT enough to guarantee identifiability. The quest for different statistics of data (higher order for instance) is therefore apparent in order to deal with this problem.

However, aside from these scattered hints, we have not, at the moment, a satisfying formulation of the concept and we are currently investigating along these lines.

5 Conclusions and Future Work

We have presented an analysis of the observability of the continuous and discrete states of a class of linear hybrid systems. We demonstrated that, under mild assumptions, one can derive necessary and sufficient conditions that the structural parameters of the model must satisfy in order to guarantee the observability of the system. Our characterization is simple and intuitive and sheds light on the geometry of the observability subspaces generated by the output of a linear hybrid system.

When the given output switches at least once in the observability interval, we derived weaker rank conditions that guarantee the uniqueness of the reconstruction of the state trajectory, even if the individual linear systems are unobservable. We believe that this is strongly related to the observability of classes of linear hybrid systems in which the switching mechanism depends on the value of the continuous state, e.g. piecewise affine systems. We expect to extend our observability results to these classes of systems in the near future.

Another important issue that we did not addressed is concerned with characterizing the set of observationally equivalent models. In linear systems theory, this is done elegantly by the Kalman decomposition, which partitions the state space into orthogonal subspaces. Future work will also address a characterization of this set for linear hybrid models.

Other aspects which remains to be investigated are the effect of measured inputs on the observability. We plan also to study the effect of "noise" on the estimation of the state and its link with some notion of distance between systems as briefly mentioned in the paper.

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